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Casimir's spheres near the Coulomb limit: energy density, pressures and radiative effects

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Abstract
We study the quantum mechanics of an infinitesimally thin spherical fluid shell of radius R, having continuous charge and mass densities e/a^2 and m/a^2 per unit area, with $a \ll R$, and subject to a Debye-type cut-off on surface-parallel wave numbers. Attention is confined to the regime $\mu \equiv 4\pi e^2 R/mc^2a^2 \ll 1$, where nonretarded (NR) Coulomb forces dominate, and the coupling to the quantized Maxwell field is only a weak perturbation. The unperturbed ground-state energy B_{NR} is of order $\hbar(e^2/ma^7)^{1/2}R^2$. Half of B_{NR} is kinetic energy, localized on the shell; the other half is Coulomb energy, with a density u appreciable only within distances of order a from the shell. The pressure $P = 3B_{\mathrm{NR}}/8\pi R^3$ follows directly from the principle of virtual work: more detailed analysis shows that $P/3$ comes from Coulomb forces, and $2P/3$ from the zero-point motion of the fluid. It seems likely that u and P behave in much the same way also for large μ. The purely Coulombic system has stable discrete-frequency excitations (plasmons); to leading order the perturbation displaces the frequencies and allows a plasmon to decay into a photon. The displacements and decay rates tally with what one infers from the exact classical multipole phase shifts, and from the already-known energy for arbitrary values of μ.

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1. Introduction and conclusions

1.1. Background

Recent work has found the Casimir binding energies B for a hydrodynamic model of hollow spherical plasma shells with radius R, developing approximations flexible enough to deliver both the dominant components governed by unretarded Coulomb interactions, and also, under appropriate conditions, the far-subdominant Boyer components, free of material constants and proportional to $\hbar c/R$. This was done by determining the exact normal modes, and then summing their zero-point energies (Barton 2004, referred to as B.III). The drawback of the method is that it does not explicitly quantize the dynamical variables (the Maxwell fields and the displacements in the plasma), and therefore affords no insight into the energy density u, nor into the forces responsible for the pressure P experienced by the shell. The Hamiltonian approach which we now adopt supplies these wants under conditions where the coupling to the quantized Maxwell fields is a weak perturbation, while Coulomb interactions continue to be treated exactly; and it leads to the crucial Debye cut-off on surface-parallel wave numbers in a way which is more conventional and perhaps more immediately persuasive. Thus the present paper is a prequel as much as a sequel to B.III, whose general comments on older work and on Boyer components will not be repeated. But it may be worth repeating, from appendix D of B.III, that roughly speaking our theory is complementary to that developed recently by Jaffe and his co-workers: theirs (unlike ours) is renormalizable in the orthodox sense, but less realistic in that it does not as yet allow for a dispersive response of material to field. For recent references see B.III, or Graham *et al* (2004).

Our own model is rooted in theories of plasmons on single base-planes in graphite (Fetter 1973) and on the giant carbon molecule C_{60} (Barton and Eberlein 1991). Some background, supporting evidence and citations are given in B.III: here we shall merely specify the system with the minimum of detail needed to make the argument self-contained.

1.2. Preliminaries

We study an infinitesimally thin spherical shell carrying a continuous fluid with mass and charge densities nm, ne per unit area, plus an immobile, uniformly distributed, overall-neutralizing background charge. The fluid displacement $\boldsymbol{\xi}$ is purely tangential. Its conjugate will be called $\boldsymbol{\Pi}$: in absence of coupling to the transverse Maxwell field, one has $\boldsymbol{\Pi} = nm\dot{\boldsymbol{\xi}}$. We assume $|\dot{\boldsymbol{\xi}}| \ll c$, so that the Lorentz force is negligible, and take the field $\boldsymbol{\xi}$ to remain small enough to allow systematic linearization in $\boldsymbol{\xi}$ and in $\dot{\boldsymbol{\xi}}$ or $\boldsymbol{\Pi}$. Then the surface charge-density reads

$$\sigma = -ne\nabla_{\|} \cdot \boldsymbol{\xi}. \tag{1.1}$$

Radial and tangential vector components will be identified by subscripts r, $\|$, but from $\boldsymbol{\xi}$ and $\boldsymbol{\Pi}$ the subscripts $\|$ are omitted as unnecessary.

Evidently, the model mimics n *delocalized* particles per unit area, call them electrons, each with charge and mass e, m. The total number is

$$N = 4\pi R^2 n. \tag{1.2}$$

The surface density n is related to some mean inter-electron distance a by

$$n \equiv 1/a^2 \tag{1.3}$$

where a, on molecules like C_{60}, is of the order of a few Bohr radii, and compares with the classical electron radius r_0 according to[1]

$$a \sim a_B \equiv \hbar^2/me^2, \qquad r_0 \equiv e^2/mc^2, \qquad x \equiv r_0/a \sim (e^2/\hbar c)^2 \simeq (1/137)^2. \tag{1.4}$$

Besides x, we also define the parameters

$$X \equiv R/a, \qquad \mu \equiv 4\pi x X = 4\pi r_0 R/a^2 = 4\pi e^2 R/mc^2 a^2. \tag{1.5}$$

[1] We use unrationalized Gaussian units: thus $e^2/\hbar c \simeq 1/137$.

As before, we consider only $N \gg 1$, which entails $X \gg 1$ (an obviously neccessary condition for any hydrodynamic model); but now we focus exclusively on $x \ll 1$. The crucial parameter is μ. In the *macroscopic scenario*, where X is so large that $\mu \gg 1$, retardation is of the essence, perturbative attempts never get off the ground, and one must use the methods of B.III. Here, by contrast, we restrict ourselves to the *molecular scenario* (with parameters roughly comparable to those of C_{60}), where $\mu \ll 1$. This scenario is approachable through the nonretarded model to be introduced presently.

We adopt the Coulomb gauge:

$$\mathbf{E} = -\nabla\Phi + \mathbf{E}_T, \qquad \mathbf{E}_T = -\dot{\mathbf{A}}/c, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \qquad \nabla \cdot \mathbf{A} = 0. \tag{1.6}$$

Then our physical system is formally defined by the minimal-coupling Hamiltonian

$$\mathcal{H} = \mathcal{H}_{\rm NR} + \mathcal{H}_{\rm int} + \mathcal{H}_{\rm rad} \tag{1.7}$$

where[2]

$$\mathcal{H}_{\rm NR} = \int \mathrm{d}S \frac{\Pi^2}{2nm} + \frac{1}{2}\int\int \mathrm{d}S\,\mathrm{d}S' \frac{\sigma(\mathbf{r})\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.8}$$

$$\mathcal{H}_{\rm rad} = \frac{1}{8\pi}\int \mathrm{d}^3 r\left\{\mathbf{E}_T^2 + \mathbf{B}^2\right\}, \tag{1.9}$$

$$\mathcal{H}_{\rm int} = \int \mathrm{d}S \left\{-\frac{e}{mc}\mathbf{\Pi}\cdot\mathbf{A}_\parallel + \frac{ne^2}{2mc^2}\mathbf{A}_\parallel^2\right\} \equiv \mathcal{H}_{\rm int,1} + \mathcal{H}_{\rm int,2}. \tag{1.10}$$

The $\int \mathrm{d}S \ldots$ are two-dimensional integrals running over the shell: $\int \mathrm{d}S \ldots \equiv R^2 \int \mathrm{d}\Omega \ldots$.

We call $\mathcal{H}_{\rm NR}$ the *nonretarded* (NR) *Hamiltonian* (reached, formally, in the limit $c \to \infty$). Taken on its own it defines the *NR model*, which would describe the shell if radiative effects were wholly negligible. Evidently $\mathcal{H}_{\rm rad}$ is the Hamiltonian for the free Maxwell field, and $\mathcal{H}_{\rm int}$ the radiative coupling. Though the NR model as such has limited application beyond fullerenes like C_{60}, it is worth attention, because it identifies important qualitative features of u and of P that are surprising, and likely to survive even when retardation is allowed for.

For comparisons with B.III, we shall generally display B in the format

$$B = (\hbar c/4\pi R)H, \tag{1.11}$$

although in a nonretarded model the dimensional prefactor $\hbar c/4\pi R$ looks out of place. Then Boyer components are identifiable as contributions to H that are pure numbers (featuring neither X nor x); this makes it obvious almost from the start that there can be no such components in the molecular scenario, where B is dominated by terms proportional to $\mu^{1/2}$, with all corrections proportional to higher powers of μ.

Finally it needs repeating from B.III that, like all pure plasma models, ours too makes B positive, because it excludes the interaction between the background ions, or the covalent bonds between the carbon atoms, which are responsible for the cohesion of real metals and of real fullerenes, respectively.

[2] The second term of (1.8) is just $\int \mathrm{d}S\,\sigma\Phi/2$, with Φ the instantaneous Coulomb potential due to σ itself. An externally applied electrostatic potential $\Phi_{\rm ext}$ would introduce a further term $\int \mathrm{d}S\,\sigma\Phi_{\rm ext}$. In our linearized model, the shell responds to such a potential exactly as would a perfectly conducting sphere having the same radius. In other words, this static response is independent of the model parameters nm and ne.

1.3. Preview and conclusions

Section 2 sets up the NR model, motivates a Debye cut-off L on the angular momenta of the normal modes (equivalent to a cut-off on their surface-parallel wave numbers), and writes down the consequent approximation B_{NR} to the zero-point energy. Section 3, the core of the paper, proceeds to the explicit quantum mechanics of the model. In section 3.1, we determine the quantized field operators $\boldsymbol{\xi}$ and $\boldsymbol{\Pi}$; and find also the equal-time commutation rules, which look elegant, but are not very helpful in calculations. The most suggestive of our results are then reported in sections 3.2 and 3.3.

Section 3.2 determines the distribution $u(r, L)$ of the Coulomb energy, and shows that for large L it is concentrated in the region $(R - a) \lesssim r \lesssim (R + a)$. Hence in the formal perfect-reflector limit $a \to 0$ (entailing $x \to \infty$), the potential energy too would become localized in the shell, as the kinetic energy automatically is in any case. The conclusion is interesting because B.III showed that B_{NR} remains the dominant part of the properly retarded energy B even in the macroscopic scenario. Thus we may be looking at one of the reasons why the dominant part of B fails to show up in older calculations, which consider only fields and energies outside the material. (See B.III for a critique of and some references to older methods.) Appendix C analyses the limit of $u(r, L)$ as $L \to \infty$ with $r \neq R$, a limit which plays no role in our theory but may help comparisons with others.

Section 3.3 uses the principle of virtual work to determine, from B_{NR}, the total pressure P experienced by the shell; and it shows that the Maxwell (here the electrostatic) stress tensor accounts only for one-third of P, the rest coming from forces exerted inside the material by the zero-point oscillations of the charge carriers. Seeing that P cannot be found from the electrostatic stresses alone, one might reasonably conjecture that the full Maxwell stress tensor would prove similarly inadequate in theories using not just $\mathcal{H}_{\mathrm{NR}}$ but the full Hamiltonian $\mathcal{H}$.

Finally, section 4 explores the effects of $\mathcal{H}_{\mathrm{int}}$ treated as a weak perturbation of a zero-order Hamiltonian[3] $\mathcal{H}_0 \equiv \mathcal{H}_{\mathrm{NR}} + \mathcal{H}_{\mathrm{rad}}$. These effects convert the discrete-frequency eigenmodes of $\mathcal{H}_{\mathrm{NR}}$ into finite-width resonances in the entirely continuous frequency spectrum of eigenmodes of the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{int}}$. More specifically, perturbation theory reproduces the widths and frequency shifts implied by the exact classical solutions already known from B.III. Details regarding this correspondence are relegated to appendix A, and the systematic approximation to the quantum shifts Δ_l to appendix B.

2. The nonretarded model

We collect some basic equations of the NR plasma model, as a preliminary to discussing its quantum mechanics in section 3, and then extending the discussion in section 4 from $\mathcal{H}_{\mathrm{NR}}$ to the full Hamiltonian $\mathcal{H}$.

2.1. Equations of motion and normal modes

The NR model admits only instantaneous Coulomb potentials Φ, whence the electric field is purely longitudinal:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{2.1}$$

[3] Risking a commonplace regarding plasmas, we stress that perturbation theory to any finite order can never capture the effects of the potential-energy (the second) term in $\mathcal{H}_{\mathrm{NR}}$. In our model, this is obvious from the mismatch of powers of n between the effective coupling-strength parameter, which is proportional to n^2, and the eigenfrequencies, which are proportional to $n^{1/2}$.

Accordingly, Newton's second law reads

$$\ddot{\boldsymbol{\xi}} = -(e/m)\nabla_{\parallel}\Phi \quad \Rightarrow \quad \boldsymbol{\xi} \equiv -\nabla_{\parallel}\Psi, \qquad \ddot{\Psi} = (e/m)\Phi, \tag{2.2}$$

with the displacement potential $\Psi(\Omega)$ defined only on the shell. Thus $\boldsymbol{\xi}$ is curl-free, and for any normal mode one has

$$\Phi_\omega \quad \text{and} \quad \Psi_\omega \propto \exp(-\mathrm{i}\omega t) \quad \Rightarrow \quad \Psi_\omega = -(e/m\omega^2)\Phi_\omega. \tag{2.3}$$

More explicitly, (2.2), (2.3) and (1.1) entail[4]

$$\boldsymbol{\xi}(\Omega) = -\nabla_{\parallel}\Psi(\Omega), \qquad \Psi_\omega(\Omega) = -(e/m\omega^2)\Phi_\omega(R,\Omega), \qquad \sigma(\Omega) = ne\nabla_{\parallel}^2\Psi(\Omega). \tag{2.4}$$

The normal modes of $\mathcal{H}_{\mathrm{NR}}$ are labelled[5] l, m (where $-l \leqslant m \leqslant l$); the frequencies are independent of m in virtue of the spherical symmetry, and will be written as ω_l. They are found (Barton and Eberlein 1991) by solving Poisson's equation for Φ_ω *off* the shell, subject to Gauss's law *on* the shell:

$$\begin{aligned} &\nabla^2\Phi(r \neq R,\Omega) = 0, && \Phi(r \to \infty,\Omega) = 0, \\ &\text{discont}\,\Phi = 0, && \text{discont}\,E_r = 4\pi\sigma \quad \Rightarrow \quad \text{discont}\,\frac{\partial\Phi_\omega}{\partial r} = \frac{4\pi ne^2}{m\omega^2}\nabla_{\parallel}^2\Phi_\omega. \end{aligned} \tag{2.5}$$

This yields

$$\left[-\frac{(l+1)R^{l+1}}{r^{l+2}} - \frac{lr^{l-1}}{R^l}\right]_{r=R} = -\frac{4\pi ne^2}{m\omega_l^2}\cdot\frac{l(l+1)}{R^2}$$

$$\Rightarrow \quad \omega_l = \sqrt{\frac{4\pi ne^2}{mR}}\hat{\omega}_l = \frac{c}{R}\mu^{1/2}\hat{\omega}_l, \qquad \hat{\omega}_l \equiv \sqrt{\frac{l(l+1)}{(2l+1)}}, \qquad (l \geqslant 1). \tag{2.6}$$

2.2. Debye cut-off

The model yields a convergent expression for B only if one imposes a cut-off $l \leqslant L$ on the normal modes. To justify it, we recall that the hydrodynamic model is meant to mimic an electron gas which in fact is granular; and then reason, as in Debye's theory of the specific heat of solids, that the number of modes should equal the number of effective degrees of freedom of the gas. Since the electrons are confined to a 2D surface, and since their motion is irrotational, there is only one degree of freedom per particle, whence the cut-off reads

$$l \leqslant L, \qquad N = 4\pi X^2 \equiv \sum_{l=1}^{L}(2l+1) = L^2 + 2L \quad \Rightarrow \quad L = \sqrt{N+1} - 1 \simeq 2\pi^{1/2}X + \cdots. \tag{2.7}$$

Another way to understand it is to note that modes with given l have tangential wave numbers of order $k_{\parallel} \sim l/R$, while a 2D gas with mean interparticle spacing a can support only waves with $ak_{\parallel} \lesssim 1$. Hence $l \lesssim R/a = X \sim L$, with the last equality from (2.7).

For much the same reasons, the shell cannot couple appreciably to photons having angular momenta $l \gtrsim L$. The calculation of radiative shifts in section 4.3 accommodates this fact

[4] On the sphere, and assuming differentiable $\boldsymbol{\Pi}(\Omega)$, it is impossible to have purely inertial flows at constant density, which would generate no NR forces apart from the constraints restricting the fluid to $r = R$. In other words there are no flows with $\nabla_{\parallel}\cdot\boldsymbol{\Pi} = 0$, and therefore no pure rotations of any kind. (Presumably, this conclusion is related to the fact that a single particle on the sphere can move only along a great circle.) One can see the impossibility by treating $\boldsymbol{\Pi}$ as the field variable in a Lagrangean $\int \mathrm{d}S\{\Pi^2/2nm - \alpha(\Omega)\nabla_{\parallel}\cdot\boldsymbol{\Pi}\}$, with α an undetermined multiplier used to enforce $\nabla_{\parallel}\cdot\boldsymbol{\Pi} = 0$. Alternatively, it can be shown that without pressure gradients (which our model excludes) there are no solutions to the nonlinear Euler equations for the velocity of the fluid.

[5] The context will prevent confusion between the m for magnetic quantum number and the m for mass.

by virtue of the selection rules; but in the fully-retarded nonperturbative calculations in B.III, where there are no discrete modes, the restriction is fundamental and is invoked right at the start, so that modes with high l never enter in the first place. In that context, and also in principle, it proves important that subject to (2.7) the theory is indeed well defined, needing no cut-offs on photon frequencies: the point is that for frequency cut-offs there is no physical justification at all.

2.3. The zero-point energy

The ground-state energy[6] is

$$B_{\rm NR} = \sum_{l=1}^{L}(2l+1)\hbar\omega_l/2 = \hbar\sqrt{(\pi ne^2/mR)}S_{\rm NR}(L) = \hbar\sqrt{N/4mR^3}S_{\rm NR}(L), \tag{2.8}$$

$$\begin{aligned} S_{\rm NR}(L) &\equiv \sum_{l=1}^{L}\sqrt{l(l+1)(2l+1)} \\ &= \sqrt{2}\left\{\frac{2}{5}L^{5/2} + L^{3/2} + \frac{7}{16}L^{1/2} - \frac{1}{32}L^{-1/2} + \mathcal{O}(L^{-3/2})\right\} + C_{\rm NR} \\ &= \frac{16\pi^{5/4}}{5}X^{5/2} + \frac{3\pi^{1/4}}{8}X^{1/2} + C_{\rm NR} + \mathcal{O}(X^{-3/2}), \qquad C_{\rm NR} \simeq -0.127, \end{aligned} \tag{2.9}$$

where the approximation in terms of L stems from the Abel–Plana formula (cf B.III). When (2.7) yields a noninteger $L \gg 1$, as generally it does, one fudges by substituting from (2.7) directly into (2.9). Thus

$$B_{\rm NR} = \hbar\sqrt{\frac{e^2}{ma^7}}\cdot R^2\cdot\frac{16\pi^{7/4}}{5}\{1+\mathcal{O}(X^{-2})\} \tag{2.10}$$

or, in our standard format,

$$B_{\rm NR} = \frac{\hbar c}{4\pi R}H_{\rm NR}, \qquad H_{\rm NR} = \mu^{1/2}\frac{32\pi^{9/4}}{5}X^{5/2}\{1+\mathcal{O}(X^{-2})\}. \tag{2.11}$$

3. Quantum mechanics of the NR model

The main point of the explicit quantization in section 3.1 is that it allows one to determine the distribution of the energy (section 3.2), and to analyse the pressure (section 3.3). As anticipated in the introduction, both results have surprising features which it is plausible to think will survive even when retardation (i.e. finite c) is allowed for. Section 3.4 merely confronts the known pressure on an undivided shell with a speculative but possibly shocking argument about the force between its two halves when it is split.

3.1. Quantization of $\mathcal{H}_{\rm NR}$, and the quantized potentials

To quantize explicitly, one diagonalizes

$$\mathcal{H}_{\rm NR} = \sum_{l=1}^{L}(\hbar\omega_l/2)\sum_{m=-l}^{l}\left\{a_{lm}^{+}a_{lm} + a_{lm}a_{lm}^{+}\right\}, \qquad \left[a_{lm}, a_{l'm'}^{+}\right] = \delta_{ll'}\delta_{mm'}, \tag{3.1}$$

[6] In the hydrodynamic model there is no Coulomb self-energy to be subtracted, because the Coulomb energy $Q^2/2R$ of a given amount of fluid vanishes at infinite dilution, i.e. as $R \to \infty$ at fixed Q.

by expanding

$$\Phi(\Omega, r, t) = \sum_{l=1}^{L} \sum_{m=-l}^{l} \sqrt{\frac{2\pi\hbar\omega_l}{(2l+1)R}} a_{lm}\, e^{-i\omega_l t} Y_{lm}(\Omega)$$
$$\times \left\{ \theta(R-r)\left(\frac{r}{R}\right)^l + \theta(r-R)\left(\frac{R}{r}\right)^{l+1} \right\} + \mathrm{Hc} \qquad (3.2)$$

where θ is the Heaviside step function, and Hc stands for Hermitean conjugate. The expansions of Ψ and thereby of $\boldsymbol{\xi}$ and $\boldsymbol{\Pi}$ then follow from (2.2). In particular

$$\Psi(\Omega, t) = -\frac{e}{m} \sum_{l=1}^{L} \sum_{m=-l}^{l} \sqrt{\frac{2\pi\hbar}{(2l+1)\omega_l^3 R}} a_{lm}\, e^{-i\omega_l t} Y_{lm}(\Omega) + \mathrm{Hc}, \qquad (3.3)$$

$$\boldsymbol{\Pi}(\Omega, t) = -ine \sum_{l=1}^{L} \sum_{m=-l}^{l} \sqrt{\frac{2\pi\hbar}{(2l+1)\omega_l R}} a_{lm}\, e^{-i\omega_l t} \nabla Y_{lm}(\Omega) + \mathrm{Hc}. \qquad (3.4)$$

The ground state $|0\rangle$ is defined by $a_{lm}|0\rangle = 0$ for all (l, m). Equations (3.2)–(3.4) indicate the time-dependence in the Heisenberg picture of the NR model, or in the interaction picture of the complete theory with $\mathcal{H}_{\mathrm{int}}$ as the perturbation.

Since the NR model makes $\boldsymbol{\xi}$ and thereby $\boldsymbol{\Pi}$ irrotational, their components are not independent degrees of freedom. This complicates the equal-time commutation rules[7], which can be expressed in many different ways. As a rule, the forms most convenient in practice are those found directly from the normal-mode expansions and from (3.1); for instance, the simplest (in fact the only tolerable) way to confirm $\boldsymbol{\Pi} = nm\dot{\boldsymbol{\xi}}$ is to use the Heisenberg equations of motion. Versions formulated in coordinate space prove far harder to use, though the eye may find them more pleasing. The most elegant form is probably the contracted relation

$$[\Psi(\Omega), \nabla' \cdot \boldsymbol{\Pi}(\Omega')] = (i\hbar/R^2) \sum_{l=1}^{L} \sum_{m=-l}^{l} Y_{lm}(\Omega) Y_{lm}^*(\Omega'). \qquad (3.5)$$

In the no-cut-off limit $L \to \infty$ this yields

$$[\Psi(\Omega), \nabla' \cdot \Pi(\Omega')] = (i\hbar/R^2)\{\delta(\Omega - \Omega') - 1/4\pi\} \qquad \text{(no cut-off)} \qquad (3.6)$$

where $-1/4\pi$ inside the braces corrects for the absence of modes with $l = 0$.

In the same limit, the full tensorial rule reads

$$C_{jj'}(\Omega, \Omega') \equiv [\xi_j(\Omega), \Pi_{j'}(\Omega')] = -\frac{i\hbar}{4\pi} \nabla_j \nabla'_{j'} D(\Omega, \Omega') \qquad \text{(no cut-off)}, \qquad (3.7)$$

$$D(\Omega, \Omega') = \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} P_l(\cos\chi), \qquad \cos\chi \equiv \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}', \qquad (3.8)$$

where $\mathbf{r} = (R, \Omega), \mathbf{r}' = (R, \Omega')$, hats denote unit vectors and the vector indices j, j' are tangential to the sphere at Ω, Ω', respectively. Splitting $(2l+1)/l(l+1) = 1/l + 1/(l+1)$, it is not too difficult to sum both terms over l (see, e.g. Barton 1989). Recombination then yields

$$D(\Omega, \Omega') = -1 + \log(2) - \log(1 - \cos\chi)$$

where only the second logarithm contributes to the derivatives:

$$C_{jj'}(\Omega, \Omega') = \frac{i\hbar}{4\pi} \nabla_j \nabla'_{j'} \log(1 - \cos\chi). \qquad (3.9)$$

[7] Similar complications beset 3D plasmas: see, e.g., Barton (1979).

To visualize this, we choose the z and the x axes so that Ω is at the North pole ($\theta = 0$), and $\Omega' = (\theta' = \chi, \phi' = 0)$. Then it becomes clear that the vector ∇' points along $\hat{\boldsymbol{\theta}}'$ at Ω', while the vector ∇ points along $-\hat{\boldsymbol{\theta}}$ at Ω: in general, the two vectors point in opposite directions along the great circle connecting Ω' and Ω. The commutators between other components of $\boldsymbol{\xi}$ and $\boldsymbol{\Pi}$ vanish. On differentiating with axes chosen as just described, the one nonzero component of the tensor turns out to read

$$C_{\theta\theta'} = \frac{\mathrm{i}\hbar}{8\pi R^2 \sin^2(\chi/2)} = \frac{\mathrm{i}\hbar}{2\pi|\mathbf{r} - \mathbf{r}'|^2}. \tag{3.10}$$

3.2. Locating the energy

Since each normal mode constitutes a simple-harmonic oscillator, the virial theorem shows that the mean kinetic energy is half the total energy, i.e. that $\langle 0| \int \mathrm{d}S\, \Pi^2/2nm|0\rangle = B_{\mathrm{NR}}/2$. Thus half of B_{NR} is automatically localized on the shell. The other half is Coulomb potential energy, distributed with a density $u(r, L) = \langle 0|(\nabla\Phi)^2/8\pi|0\rangle$. *We proceed to demonstrate that for* $L \gg 1$ *it is, roughly speaking, confined to distances of order a from the shell.*

To study u, we use (3.2) for Φ, exploit isotropy to write $u = \int \mathrm{d}\Omega\, u/4\pi$, and after an integration by parts find straightforwardly that[8]

$$\begin{aligned} u(r, L) &= \frac{1}{8\pi}\int \frac{\mathrm{d}\Omega}{4\pi}\langle 0|\nabla\Phi\cdot\nabla\Phi|0\rangle = \frac{1}{8\pi}\int\frac{\mathrm{d}\Omega}{4\pi}\langle 0|\left\{(\nabla_r\Phi)^2 + \frac{l(l+1)}{r^2}\Phi^2\right\}|0\rangle \\ &= \frac{1}{16\pi R^3}\sum_{l=1}^{L}\hbar\omega_l(2l+1)\left\{\theta(R-r)l\left(\frac{r}{R}\right)^{2l-2} + \theta(r-R)(l+1)\left(\frac{R}{r}\right)^{2l+4}\right\}. \end{aligned} \tag{3.11}$$

This expression is easier to appreciate in terms of dimensionless variables featuring the energy β per unit surface area, and distances measured from the shell in units of a:

$$\beta(L) \equiv B_{\mathrm{NR}}/4\pi R^2, \qquad \xi \equiv |r - R|/a, \qquad \rho(\xi, L) \equiv au(r, L)/\beta(L), \tag{3.12}$$

$$\begin{aligned} &\rho(\xi, L) \\ &= \frac{(1/2L)\sum_{l=1}^{L}(2l+1)\hat{\omega}_l\{\theta(R-r)l(1-\xi/L)^{2l-2} + \theta(r-R)(l+1)(1+\xi/L)^{-2l-4}\}}{\sum_{l=1}^{L}(2l+1)\hat{\omega}_l}. \end{aligned} \tag{3.13}$$

Through L, this expression depends on X as well as on ξ; but for $L \gg 1$ it simplifies very considerably. In that case both the numerator and the denominator are dominated by large $l \lesssim L$. Accordingly we approximate

$$(1 \pm \xi/L) = \exp\log(1 \pm \xi/L) \simeq \exp(\pm\xi/L); \tag{3.14}$$

keep factors $\exp(\pm l\xi/L)$ but set $\exp(\pm\xi/L) \to 1$; in the denominator replace $\sum_l \to \int \mathrm{d}l$; and find eventually that in the limit (where numerator and denominator both diverge) $\rho(\xi) \equiv \rho(\xi, \infty)$ is the same monotonically decreasing positive function of ξ on both sides of the shell[9]:

[8] Curiously, $u(r = 0)$ is nonzero: the mean-square electric field fails to vanish at the origin.

[9] In the limit the left-hand side of $\int_0^\infty \mathrm{d}r\, r^2[u] = \beta R^2/2$ reduces to $2R^2\int_0^\infty a\,\mathrm{d}\xi[\beta\rho(\xi)/a] = 2\beta R^2\int_0^\infty \mathrm{d}\xi\,\rho(\xi)$, whence (3.15) should satisfy the norming condition $\int_0^\infty \mathrm{d}\xi\,\rho(\xi) = 1/4$, as indeed it does.

$$\rho(\xi, L \gg 1) \simeq \rho(\xi) \equiv \lim_{L\to\infty} \rho(\xi, L) = \lim_{L\to\infty} \frac{5}{4L^{7/2}} \sum_{l=1}^{L} l^{5/2} \exp(-2l\xi/L)$$
$$= \frac{5}{512}\left\{15\sqrt{2\pi}\xi^{-7/2}\mathrm{erf}(\sqrt{2\xi}) - \mathrm{e}^{-2\xi}\left[\frac{64}{\xi} + \frac{80}{\xi^2} + \frac{60}{\xi^3}\right]\right\}, \tag{3.15}$$

$$\rho(\xi \ll 1) = \frac{5}{14} - \frac{5\xi}{9} + \frac{5\xi^2}{11} + \cdots, \qquad \rho(\xi \gg 1) = \frac{75\sqrt{2\pi}}{512}\xi^{-7/2} + \mathcal{O}\left(\frac{\mathrm{e}^{-2\xi}}{\xi}\right). \tag{3.16}$$

The mere fact that the limit is well defined makes it a function of ξ alone: this suffices to verify that the energy density is indeed localized in the way described above. (Moreover, with increasing L the approximation $\rho(\xi)$ approaches the true $\rho(\xi, L)$ quite fast. For instance, as r rises from 0 to R, numerical evaluation shows that $\rho(\xi)/\rho(\xi, 10)$ falls from 0.969 to 0.893, while $\rho(\xi)/\rho(\xi, 100)$ falls from 0.997 only to 0.989.)

Finally, for comparison with other theories, appendix C steps outside our own to consider the so-called no-cut-off limit $u_\infty(r) \equiv \lim_{L\to\infty} u(r, L)$. This is a very different thing from the scaled function $\rho(\xi)$ used above as a mere approximation to $\rho(\xi, L \gg 1)$: the definition (3.12) shows that $\rho(\xi)$ is the limit $L \to \infty$ of the ratio $u(r, L)/\beta(L)$, whereas the ratio of the limits $u(r, L \to \infty)/\beta(L \to \infty)$ vanishes in the sense that $\beta(L \to \infty)$ diverges.

3.3. Pressures and forces

The principle of virtual work equates the change in[10] B to the work done by the total pressure P in small virtual variations of R:

$$P = -\frac{1}{4\pi R^2}\left(\frac{\partial B}{\partial R}\right)_N = -\frac{1}{4\pi R^2}\left(\frac{\partial B}{\partial R}\right)_L = -\frac{1}{4\pi R^2}\left(-\frac{3B}{2R}\right) = \frac{3B}{8\pi R^3}. \tag{3.17}$$

The third step is obvious from (2.8).

The total force between say the northern and southern hemispheres is a repulsion

$$F = \pi R^2 P. \tag{3.18}$$

One can understand this either as the sum of the northward components of the forces acting on all infinitesimally small surface elements of the northern hemisphere, or more easily as the total force that would be exerted across the equatorial plane by an excess pressure P in the interior.

To derive (3.17) *directly rather than through the principle of virtual work, one must first realize that there are two quite different contributions to P, one electrostatic and the other mechanical.*

The outward *electrostatic force* per unit area is $\sigma E_{r,\mathrm{ave}}$, where

$$\sigma = \{E_r(R+) - E_r(R-)\}/4\pi, \qquad E_{r,\mathrm{ave}} \equiv \{E_r(R+) + E_r(R-)\}/2.$$

Thus the electrostatic pressure reads

$$P_{\mathrm{es}} = \langle 0|\sigma E_{r,\mathrm{ave}}|0\rangle = \langle 0|\left\{E_r^2(R+) - E_r^2(R-)\right\}\big/8\pi|0\rangle, \tag{3.19}$$

as expected from the jump in the rr component $\left\{E_r^2 - \mathbf{E}_\parallel^2\right\}\big/8\pi$ of the electrostatic part of the Maxwell stress tensor, given that $\mathbf{E}_\parallel^2$ is continuous. Since $\mathbf{E} = -\nabla\Phi$, the vacuum

[10] To ease the typography, section 3.3 omits the subscript NR that B and P ought to carry.

expectation-value follows readily from (3.2):

$$P_{\rm es} = \frac{1}{8\pi}\sum_{l=1}^{L}\left[\frac{2\pi\hbar\omega_l}{(2l+1)R}\right]\frac{1}{R^2}[(l+1)^2 - l^2]\sum_{m=-l}^{l}|Y_{lm}(\Omega)|^2$$
$$= (1/16\pi R^3)\sum_{l=1}^{L}(2l+1)\hbar\omega_l = B/8\pi R^3. \qquad (3.20)$$

In addition to $P_{\rm es}$ there is also an outward *mechanical force* per unit area, stemming from the surface tension Σ of the fluid. To identify it we recall that the pressure of an ideal 3D gas is $\nu m v^2/3$, with v^2 the mean-square speed of the particles, and ν the number per unit volume. The 2D analogue of this formula yields a tangential stress in the shell (a force across unit length), related to the surface tension by

$$-\Sigma = nm\dot{\boldsymbol{\xi}}^2/2 = \mathbf{\Pi}^2/2nm. \qquad (3.21)$$

On any element of the shell this produces the same force as would an excess pressure $-2\Sigma/R$, by the argument familiar from soap bubbles. (Again, consider the northern hemisphere. It experiences a net southward force from the surface tension acting across the equator. If this force is ascribed instead to an excess pressure in the interior, then, by the same argument as we used for $P_{\rm es}$, one has $2\pi R \times$ (surface tension) $= \pi R^2 \times$ (excess pressure).) Accordingly

$$P_{\rm mech} = (2/R)\langle 0|(-\Sigma)|0\rangle = B/4\pi R^3. \qquad (3.22)$$

In view of (3.21), the expectation value follows from the virial theorem and from isotropy by the same argument as was used in section 3.2: $\langle 0|(-\Sigma)|0\rangle = (B/2)/4\pi R^2$.

Combining, we obtain the *total pressure*

$$P = P_{\rm es} + P_{\rm mech} = 3B/8\pi R^3, \qquad (3.23)$$

conformably with (3.17).

Regarding its operational significance, we recall that P stems from the positive energy B due directly to the charged fluid, which mimics only the conduction electrons. To stabilize the structure there must be other forces, attractive at long range to stop it from disintegrating, and repulsive at short range to stop it from collapsing. Thus one might try to elaborate the model as a thin shell made of ordinary elastic material with given elastic moduli, and ask how it would react when P is switched on (see, e.g., Landau and Lifshitz (1986)).

3.4. The force between severed hemispheres?

What happens to the force F if the sphere is cut along the equator, without appreciably separating the hemispheres?

Cutting entails boundary conditions on the north–south component of the fluid displacements $\boldsymbol{\xi}$ along the cut, and allows line charges to build up there. Hence it rearranges the normal modes, changes their frequencies and presumably eliminates from F any contribution like $\pi R^2 P_{\rm mech}$. In short (and by contrast to insulators), the cutting process certainly affects the forces. Pending detailed calculations, which would be very laborious, it is tempting to assert, as an *almost theorem*, that F becomes attractive. An *almost proof* runs as follows. (i) Each hemisphere by itself effectively consists of a set of normal-mode oscillators; (ii) the Coulomb interaction between the hemispheres is bilinear in the creation/annihilation operators, whence it has zero expectation value in the uncoupled direct-product ground state; (iii) therefore it lowers the energy (proved variationally); whence (iv) the coupling should become weaker and the ground-state energy should rise (become less negative) as the hemispheres separate.

Of course this particular argument fails for separations $\gtrsim 2\pi c/\omega_1$, where retardation becomes essential.

4. Radiative coupling: widths and shifts

We explore the effects of $\mathcal{H}_{\text{int}}$ treated perturbatively. They turn the discrete-frequency modes of the NR model into the lowest *TM* resonances in the (wholly continuous) frequency spectrum of the true Maxwellian system. It is best to start by quantizing the free electromagnetic field in multipolar form, as in section 4.1. The transition rates w_l are then found in section 4.2, and the energy shifts Δ_l in section 4.3. They are shown to reproduce, under appropriate conditions, the widths γ_l of the resonances, and their frequency shifts $\Delta\omega_l$ away from ω_l, as determined from the classically calculated phase shifts. Section 4.3 also shows that the dominant component of the total second-order quantal energy shift tallies with the corresponding part of B found nonperturbatively in B.III. The systematics of the Δ_l are relegated to appendix B.

4.1. Multipolar quantization of the free Maxwell field

Recall that we work in the Coulomb gauge[11], with (1.6).

We shall need the standard scalar multipole amplitudes

$$\varphi_{lmk}(\mathbf{r}) \equiv i^l\sqrt{\frac{2}{\pi}}\, j_l(kr)Y_{lm}(\Omega), \qquad \nabla^2\varphi_{lmk} = -k^2\varphi_{lmk}, \tag{4.1}$$

$$\mathbf{L} \equiv -\mathrm{i}\mathbf{r}\times\nabla, \qquad L^2\varphi_{lmk} = l(l+1)\varphi_{lmk}, \qquad L_z\varphi_{lmk} = m\varphi_{lmk},$$
$$\int \mathrm{d}^3r\, \varphi^*_{lmk}(\mathbf{r})\varphi_{l'm'k'}(\mathbf{r}) = \delta_{ll'}\delta_{mm'}\delta(k-k')/kk'. \tag{4.2}$$

The vector potentials of the individual multipole fields are given in terms of the φ_{lmk} by Bouwkamp and Casimir (1954) (see also Jackson (1975), section 16.2). We change their labels e, m to *TM*, *TE*, and define

$$\mathbf{A}^{TE}_{lmk} \equiv \frac{1}{\sqrt{l(l+1)}}\mathbf{L}\varphi_{lmk}, \qquad \mathbf{A}^{TM}_{lmk} \equiv \frac{1}{k}\nabla\times\mathbf{A}^{TE}_{lmk}, \qquad (l \geqslant 1), \tag{4.3}$$

so that

$$\int \mathrm{d}^3r\, \mathbf{A}^{s*}_{lmk}\cdot\mathbf{A}^{s'}_{l'm'k'} = \delta_{ss'}\delta_{ll'}\delta_{mm'}\delta(k-k')/kk', \qquad s = TE, TM. \tag{4.4}$$

To separate the tangential from the radial components of $\mathbf{A}^{TM}_{lmk}$, one can use equation (16.49) from Jackson (1975):

$$\mathrm{i}\nabla\times\mathbf{L} = \mathbf{r}\nabla^2 - \nabla(1+\mathbf{r}\cdot\nabla) = \mathbf{r}\nabla^2 - \nabla\frac{\partial}{\partial r}r$$
$$\Rightarrow \quad \left(\mathbf{A}^{TM}_{lmk}\right)_\parallel = \sqrt{\frac{2}{\pi}}\,\tilde{j}'_l(kr)\frac{\mathrm{i}}{k\sqrt{l(l+1)}}\nabla Y_{lm} \tag{4.5}$$

where the Riccati–Bessel functions are defined by

$$\tilde{j}_l(z) \equiv zj_l(z) \tag{4.6}$$

and the prime on $\tilde{j}'_l(kr)$ signifies the derivative with respect to the argument kr; and similarly for $\tilde{y}_l$.

To expand $\mathbf{A}$ in terms of the $\mathbf{A}^s_{lmk}$, one chooses coefficients b so as to ensure that the free Maxwell Hamiltonian assumes the standard form

$$\mathcal{H}_{\text{rad}} = \sum_s\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\int_0^\infty \mathrm{d}k\, k^2\frac{\hbar ck}{2}\left\{b^{(s)+}_{lmk}b^{(s)}_{lmk} + b^{(s)}_{lmk}b^{(s)+}_{lmk}\right\} \tag{4.7}$$

[11] By contrast, B.III gave the exact normal modes in terms of their $\mathbf{B}$ and of their *total* $\mathbf{E}$ fields: and the total $\mathbf{E}$ field of *TM* modes is not divergence-free (i.e. not purely transverse), because it has a discontinuity across the shell as dictated by Gauss's law.

where

$$\left[b^{(s)}_{lmk} b^{(s')+}_{l'm'k'} - b^{(s')+}_{l'm'k'} b^{(s)}_{lmk}\right] = \delta_{ss'}\delta_{ll'}\delta_{mm'}\delta(k-k')/kk'. \tag{4.8}$$

This is achieved by

$$\mathbf{A}(\mathbf{r},t) = \sum_s \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \int_0^\infty \mathrm{d}k\, \sqrt{2\pi\hbar c k^3}\, b^{(s)}_{lmk}\, \mathrm{e}^{-\mathrm{i}ckt}\, \mathbf{A}^s_{lmk}(\mathbf{r}) + \mathrm{Hc}. \tag{4.9}$$

4.2. Widths

The linear coupling

$$\mathcal{H}_{\mathrm{int},1} = -(e/mc)\int \mathrm{d}S\, \mathbf{\Pi}\cdot\mathbf{A}_\parallel \tag{4.10}$$

causes the one-plasmon state $a^+_{lm}|0\rangle$ to decay into one-*TM*-photon states $b^{TM+}_{lmk}|0\rangle$. The matrix element responsible reads

$$\mathcal{M}_{lk} \equiv \langle 0|b^{TM}_{lmk}\mathcal{H}_{\mathrm{int},1}a^+_{lm}|0\rangle = \frac{\mathrm{i}ne^2R^2}{mc}\left[\frac{2\pi\hbar c}{k}\cdot\frac{2\pi\hbar}{(2l+1)\omega_l R}\right]^{1/2}\int \mathrm{d}\Omega\, \mathbf{A}^{TM*}_{lmk}(R,\Omega)\cdot\nabla Y_{lm}$$
$$= \frac{2\pi\hbar n e^2}{m}\left[\frac{l(l+1)}{(2l+1)\omega_l c R k^3}\right]^{1/2}\sqrt{\frac{2}{\pi}}\,\tilde{j}'_l(kR) \tag{4.11}$$

where the last line follows after some manipulation involving an integration by parts. The selection rules on $\mathcal{M}_{lk}$ stem from spherical symmetry and reflection invariance (conservation of angular momentum and of parity): it is easily verified that the analogous matrix element with a *TE* photon would vanish.

The Golden Rule of time-dependent perturbation theory then gives the decay rate

$$w_l = \frac{2\pi}{\hbar}\int_0^\infty \mathrm{d}k\, k^2\delta(\hbar\omega_l - \hbar ck)|\mathcal{M}_{lk}|^2 = \frac{4\pi ne^2}{mc}\tilde{j}'^2_l(K_lR) = \frac{c\mu}{R}\tilde{j}'^2_l(K_lR), \qquad K_l \equiv \omega_l/c. \tag{4.12}$$

Finally, in the molecular scenario where $\mu \ll 1$, we see that

$$K_lR = \frac{\omega_l R}{c} = \hat{\omega}_l\sqrt{\mu} \ll 1 \quad\Rightarrow\quad \tilde{j}'^2_l(K_lR) \simeq \frac{(l+1)^2}{(2l+1)!!^2}(K_lR)^{2l} \tag{4.13}$$

whence

$$w_l \simeq \frac{c}{R}\cdot\frac{(l+1)^2}{(2l+1)!!^2}\cdot\hat{\omega}_l^{2l}\mu^{l+1}. \tag{4.14}$$

Appendix A verifies that w_l agrees with the value inferred from the width of the lowest *TM* resonance as determined directly from the phase shift, i.e. without appeal to quantum mechanics.

4.3. Shifts

The interaction with the quantized Maxwell field changes the unperturbed ground-state energy $\hbar\omega_l/2$ of each NR normal-mode oscillator. We consider the change only to order e^2. As explained in B.III, the perturbation $\langle 0|\mathcal{H}_{\mathrm{int},2}|0\rangle$ is irrelevant to B, because it is just the self-energy of the charge carriers; hence we consider only the shifts Δ_l due to $\mathcal{H}_{\mathrm{int},1}$. In virtue of the selection rules, Δ_l features only the intermediate states $|\rangle \equiv a^{+}_{l,-m} b^{TM+}_{lmk}|0\rangle$, and it is easily seen that $|\langle|\mathcal{H}_{\text{int},1}|0\rangle|^2 = |\mathcal{M}_{lk}|^2$. Hence second-order perturbation theory yields

$$\Delta_l = -\int_0^\infty \mathrm{d}k\, k^2 \frac{|\mathcal{M}_{lk}|^2}{\hbar[\omega_l + ck]} = -\frac{\hbar c}{4\pi R} \cdot 2\mu^{3/2}\hat{\omega}_l \mathcal{J}_l(\eta_l), \tag{4.15}$$

$$\mathcal{J}_l(\eta) \equiv \int_0^\infty \frac{\mathrm{d}x}{x(x+\eta)} \tilde{j}_l^{\prime 2}(x), \qquad \eta_l \equiv \frac{R\omega_l}{c} = \mu^{1/2}\hat{\omega}_l. \tag{4.16}$$

An asymptotic expansion when $\eta \ll 1$ is derived in appendix B. Here we keep only the dominant term, obtained by setting $\eta = 0$:

$$\Delta_l \simeq -\frac{\hbar c}{4\pi R}\mu^{3/2}\pi\hat{\omega}_l \mathcal{Q}_l, \qquad \mathcal{Q}_l \equiv \frac{(2l^2+2l+3)}{(2l-1)(2l+1)(2l+3)}, \tag{4.17}$$

$$\frac{\Delta_l}{(\hbar\omega_l/2)} \simeq -\frac{1}{2}\mu\mathcal{Q}_l. \tag{4.18}$$

The ratio reminds one that perturbation theory works only if $\mu \ll 1$.

The shift Δ_l may be compared with two other results found very differently.

First, equation (A.8) in appendix A shows that $\Delta_l/\hbar$ from (4.17) equals $\Delta\omega_l/2$, where $\Delta\omega_l$ is the difference between ω_l and the lowest *TM* resonance frequency as determined directly from the phase shifts.

Second, B.III supplies the fully retarded contributions B_l to B from each (l, m), albeit only in an approximation which is unwarranted unless $l \gg 1$ (these being the values which dominate B when $R \gg a$). The results to be compared with (4.17) are those for small μ, namely

$$B_l = (\hbar c/2\pi R)F_l(\alpha), \qquad \alpha \equiv \mu/(2l+1) \ll 1.$$

Equations (4.22) and (6.3) of B.III give

$$B_l \simeq \frac{\hbar c}{4R}(2l+1)\left[\frac{\mu}{(2l+1)}\right]^{1/2}\left\{1 - \frac{1}{4}\left[\frac{\mu}{(2l+1)}\right] + \cdots\right\}, \tag{4.19}$$

both terms stemming wholly from *TM* modes. For $l \gg 1$ the first term reduces as it should to $\hbar\omega_l/2$, and the ratio of the second to the first term becomes $-\mu/8l$, the same as the ratio dictated by (4.18) with $\mathcal{Q}_l \sim 1/4l$.

Finally, the total radiative shift

$$\Delta B \equiv \sum_{l=1}^{L}(2l+1)\Delta_l \simeq -\frac{\hbar c}{4\pi R} \cdot \frac{2\pi^{7/4}}{3}\mu^{3/2}X^{3/2} \tag{4.20}$$

follows from (4.17) on approximating $\sum_l \ldots \to \int^L \mathrm{d}l \ldots$ for $L \gg 1$. Comparison with (2.11) then yields

$$\Delta B/B_{\text{NR}} \simeq -(5/48\pi^{1/2})\mu X^{-1}. \tag{4.21}$$

Appendix A. Classical widths and frequency shifts from the Jost functions

We determine the frequencies and widths of the lowest resonances of the classical *TM* phase shifts, and find that they tally, when they should, with quantum-mechanical results from section 4 and from B.III.

The coupling to the Maxwell field dissolves the (l, m) plasmon mode of the NR model (sharp frequency ω_l) in the continuum of transverse-magnetic (l, m) photons, where it generates the lowest resonance. The resonance corresponds to the pole of the S matrix due to the zero of the Jost function $f_l^{TM}(-q)$ nearest the origin in the lower-half complex q plane, say at $q = q_1 - \mathrm{i}q_2$:

$$f_l^{TM}(-q) = 1 + \mathrm{i}\frac{\mu}{q}\tilde{j}_l'(q)\tilde{h}_l^{(1)'}(q) = 0 \quad \Rightarrow \quad q_1 - \mathrm{i}q_2 - \mu\tilde{j}_l'(q)\tilde{y}_l'(q) + \mathrm{i}\mu\tilde{j}_l'^2(q) = 0. \tag{A.1}$$

The Jost function is quoted from B.III; we recall that $q \equiv kR = \omega R/c$, and suppress the index l that q ought to carry. The frequency shift $\Delta\omega_l$ and the width γ_l are given by

$$\omega_l + \Delta\omega_l = cq_1/R, \qquad \gamma_l = cq_2/R. \tag{A.2}$$

Small μ entails small $|q|$, and one expands $\tilde{j}_l'(q)$ and $\tilde{y}_l'(q)$ accordingly:

$$\tilde{j}_l'(q) = \frac{1}{(2l+1)!!}\left\{(l+1)q^l - \frac{(l+3)}{2(2l+3)}q^{l+2} + \cdots\right\}, \tag{A.3}$$

$$\tilde{y}_l'(q) = (2l-1)!!\left\{\frac{l}{q^{l+1}} + \frac{(l-2)}{2(2l-1)}\cdot\frac{1}{q^{l-1}} + \cdots\right\}. \tag{A.4}$$

Consistency then allows at most two terms in each of the products $\tilde{j}_l'\tilde{y}_l'$ and $\tilde{j}_l'^2$, reducing (A.1) to

$$(1 + \mu\mathcal{Q}_l)q^2 - \mu\hat{\omega}_l^2 + \mathrm{i}\mu\frac{1}{(2l+1)!!^2}\left\{(l+1)^2q^{2l+1} - \frac{(l+1)(l+3)}{(2l+3)}q^{2l+3}\right\} \simeq 0, \tag{A.5}$$

where $\mathcal{Q}_l$ enters through an apparently fortuitous combination of coefficients originating from (A.3) and (A.4), rather than as the integral (B.2) already quoted in (4.17).

To the accuracies we want one can proceed by successive approximations, $q = q^{(0)} + q^{(1)} + \cdots$. Leading order merely confirms that $q_1^{(0)} = \mu^{1/2}\hat{\omega}_l = \omega_l R/c, q_2^{(0)} = 0$. Next one finds

$$\gamma_l \simeq \frac{c}{R}q_2^{(1)} \simeq \frac{2\pi ne^2}{mc}\left[\frac{(l+1)}{(2l+1)!!}\right]^2\hat{\omega}_l^{2l}\mu^l. \tag{A.6}$$

Thus $2\gamma_l$ agrees as it should with the decay rate w_l from (4.14).

Finally, the first two terms of (A.5) yield q_1 through

$$\left(q_1^{(0)} + q_1^{(1)}\right)^2 \simeq \mu\hat{\omega}_l^2\big/(1 + \mu\mathcal{Q}_l) \quad \Rightarrow \quad q_1^{(1)} \simeq -\mu^{3/2}\hat{\omega}_l\mathcal{Q}_l/2. \tag{A.7}$$

Then

$$\Delta\omega_l \simeq (c/R)q_1^{(1)} = -(c\mu^{3/2}/2R)\hat{\omega}_l\mathcal{Q}_l \tag{A.8}$$

tallies with the perturbative shift Δ_l from (4.17), in the sense that $\hbar\Delta\omega_l/2 = \Delta_l$. The agreement is welcome insurance against calculational errors, since the two methods follow very different paths to the quite un-intuitive combination $\mathcal{Q}_l$.

Appendix B. Systematics of the shifts Δ_l

For $\eta_l \ll 1$, an asymptotic approximation to $\mathcal{J}_l(\eta)$ in (4.15) and (4.16) emerges if one expands its integrand by powers of η/x as far as one can without incurring a divergence at the lower limit:

$$\mathcal{J}_l(\eta) = \int_0^\infty \frac{\mathrm{d}x}{x(x+\eta)}\tilde{j}_l'^2(x) = \int_0^\infty \frac{\mathrm{d}x}{x^2}\tilde{j}_l'^2\left\{1 - \frac{\eta}{x} + \frac{\eta^2}{x^2} + \cdots\right\}. \tag{B.1}$$

This admits one term when $l = 1$, three terms when $l \geqslant 2$ and so on. To find the general term one integrates by parts (repeatedly), and eliminates some second derivatives through $j_l'' = -2j_l'/x - j_l + l(l+1)j_l/x^2$. Eventually one obtains

$$\int_0^\infty \frac{\mathrm{d}x}{x^n}\tilde{j}_l'^2 = \left\{\frac{1}{2}n(n+1) - l(l+1)\right\}\int_0^\infty \frac{\mathrm{d}x}{x^n}j_l^2 + \int_0^\infty \frac{\mathrm{d}x}{x^{n-2}}j_l^2,$$

featuring the standard integrals

$$\int_0^\infty \frac{\mathrm{d}x}{x^p}j_l^2 = \frac{\pi\Gamma(p+1)\Gamma(l+1/2-p/2)}{2^{p+2}\Gamma^2(1+p/2)\Gamma(l+3/2+p/2)}.$$

Then the first and second terms of (B.1), convergent respectively for $l \geqslant 1$ and for $l \geqslant 2$, reduce to

$$\int_0^\infty \frac{\mathrm{d}x}{x^2}\tilde{j}_l'^2 = \frac{\pi}{2}\mathcal{Q}_l, \qquad \int_0^\infty \frac{\mathrm{d}x}{x^3}\tilde{j}_l'^2 = \frac{1}{6}\mathcal{P}_l, \tag{B.2}$$

$$\mathcal{Q}_l \equiv \frac{(2l^2+2l+3)}{(2l-1)(2l+1)(2l+3)}, \qquad \mathcal{P}_l \equiv \frac{(l^2+l+6)}{(l-1)l(l+1)(l+2)}, \tag{B.3}$$

with the $\mathcal{Q}_l$ already quoted in (4.17). But for $l = 1$, the second term must be found by evaluating $\mathcal{J}_1(\eta)$ exactly (in terms of sine and cosine integrals) before approximating it for small η. This leads to

$$\mathcal{J}_1(\eta) = \frac{7\pi}{30} - \eta\left(\frac{4}{9}\log\left[\frac{1}{2\eta}\right] - \frac{4}{9}\mathcal{E} + \frac{23}{36}\right) + \mathcal{O}(\eta^2)$$

where $\mathcal{E}$ is Euler's constant. From these results one constructs

$$\Delta_1 = -\frac{\hbar c}{4\pi R}\cdot 2\mu^{3/2}\hat{\omega}_1\left\{\frac{7\pi}{30} + \mu^{1/2}\hat{\omega}_1\left(\frac{4}{9}\log\left[\frac{1}{2\mu^{1/2}\hat{\omega}_1}\right] - \frac{4}{9}\mathcal{E} + \frac{23}{36}\right) + \mathcal{O}\left(\mu\hat{\omega}_1^2\right)\right\}, \tag{B.4}$$

$$\Delta_l = -\frac{\hbar c}{4\pi R}\cdot 2\mu^{3/2}\hat{\omega}_l\left\{\frac{\pi}{2}\mathcal{Q}_l - \mu^{1/2}\hat{\omega}_l\frac{1}{6}\mathcal{P}_l + \mathcal{O}(\mu\hat{\omega}_l^2)\right\}, \qquad (l \geqslant 2). \tag{B.5}$$

As regards orders of magnitude, let us ignore powers of X, and take $x \equiv r_0/a \sim (e^2/\hbar c)^2$, as in (1.4). Then (4.18) shows that the ratio of the first term of Δ_l to $\hbar\omega_l$ is of order $\mu \sim x \sim (e^2/\hbar c)^2$. In atomic physics this is the relative order of magnitude of the fine-structure splitting, i.e. of the corrections for retardation and for the relativistic variation of inertial mass with velocity. The second term of Δ_l is smaller by a further factor of $\mu^{1/2} \sim x^{1/2} \sim e^2/\hbar c$; this makes it comparable to the Lamb shift, and calls for two final comments.

First, pending further analysis it is unclear whether our accuracy allows one to keep the second and higher terms of (B.1), because comparable contributions might arise from higher orders of perturbation theory, e.g., to fourth order in $\mathcal{H}_{\mathrm{int},1}$, or jointly to second order in $\mathcal{H}_{\mathrm{int},1}$ and first order in $\mathcal{H}_{\mathrm{int},2}$. Second, structurally speaking the first term (with $\mathcal{J}_l(\eta_l) \to \mathcal{J}_l(0)$, as in section 4.2) bears a resemblance to the component which the Bethe theory drops from the atomic Lamb shift, on the grounds that it is cancelled by mass renormalization. (For the standard theory, see e.g., Bethe and Salpeter (1957); for versions that admit retardation, see, e.g., Grotch (1981), and also Au and Feinberg (1974).) The analogy lies in the fact that the denominator of this term is independent of the plasmon excitation-energy $\hbar\omega_l$. On the other hand, the mere analogy is inconclusive, because the mass–density parameter nm in the hydrodynamic model plays a role somewhat different from that of the electron mass in atomic structure. For instance, in the absence of purely inertial flows on the sphere (cf footnote 4) it is unclear how nm should be defined operationally (one may need to compare the sphere with a plane); and the difference respecting Coulomb self-energies likewise inspires caution (cf footnote 6). Meanwhile, the writer suspects that in fact the analogy is false; but if it is not, then to preserve the correspondence noted in section 4.3 between quantum and classical shifts one would have to find mutually compatible mass-renormalization prescriptions in both domains.

Appendix C. The energy density in the no-cut-off limit

In the spirit explained at the end of section 3.2 (and following the suggestion of an anonymous referee), we take a brief look at

$$u_\infty(r) \equiv \lim_{L\to\infty} u(r, L), \qquad r \neq R,$$

commonly called the no-cut-off limit of u. All approaches agree that this limit is finite: the crucial differences lie in whether, and how, they try to link B with integrals over $u_\infty(r)$, seeing that prima facie $u_\infty(r)$ always diverges nonintegrably as $r \to R$. We stress that in our model the question is irrelevant, because the model gives B not through such an integral, but (in the NR limit) directly as in section 2.3. A careful discussion of the role of $u_\infty(r)$ in older approaches is given by Candelas (1982), and in the modern renormalizable theory by Graham *et al* (2002, 2003).

Bearing these provisos in mind, it is easy write the limit of (3.11) as

$$u_\infty(r) = \frac{\hbar}{16\pi R^3}\sqrt{\frac{4\pi e^2}{mRa^2}}\{\theta(R-r)U_{\rm in} + \theta(r-R)U_{\rm out}\},$$

$$U_{\rm in} = \sum_{l=1}^{\infty}\sqrt{l^3(l+1)(2l+1)}z^{l-2}, \qquad U_{\rm out} = \sum_{l=1}^{\infty}\sqrt{l(l+1)^3(2l+1)}/z^{2l+4}, \qquad z \equiv r/R.$$

Far from the shell, i.e. for $z \ll 1$ or $z \gg 1$, the power series $U_{\rm in}, U_{\rm out}$ converge straightforwardly. Just inside/outside the shell, i.e. for $z \sim 1$, we approximate as follows. (i) Expand the square roots by falling powers of l : $\sqrt{\ldots} = \sum_{p=0}^{\infty} c_p l^{5/2-p}$. (ii) Split each $U = U^{(1)} + U^{(2)}$, so that $U^{(1)}$ contains those sums over l that would diverge when $z = 1$, namely those with $p = 0, 1, 2, 3$. (iii) In $U^{(1)}$, approximate each of the four $\sum_l$ .. through the Euler–Maclaurin formula truncated to

$$\sum_{l=1}^{\infty} f(l) \simeq \int_1^\infty {\rm d}l\, f + f(1)/2 - f'(1)/12 + f'''(1)/120$$

where $f(l)$ is $l^p z^{2l-2}$ in $U_{\rm in}$ and l^p/z^{2l+4} in $U_{\rm out}$. The integrals feature Whittaker functions, whose asymptotics fortunately turn out to be quite simple. (iv) For $z \lessgtr 1$, set $z = 1 \mp y$, and expand asymptotically for small y, keeping terms that would diverge or remain finite as $y \to 0+$, but dropping terms that would vanish. (v) In $U^{(2)}$, set $z = 1$ and evaluate it numerically as the convergent sum $\sum_{l=1}^{\infty}\left[\sqrt{\ldots} - \sum_{p=0}^{3} c_p l^{5/2-p}\right]$. (vi) Combine this approximation to $U^{(2)}$ with the finite term from step (iv). The end-results read

$$\left.\begin{matrix}U_{\rm in}\\ U_{\rm out}\end{matrix}\right\} = \frac{15\sqrt{\pi}}{64}y^{-7/2} \pm \frac{51\sqrt{\pi}}{256}y^{-5/2} + \frac{345\sqrt{\pi}}{2048}y^{-3/2} \pm \frac{1563\sqrt{\pi}}{8192}y^{-1/2}$$
$$- \left\{\begin{matrix}0.104\\ 0.246\end{matrix}\right\} + (\text{terms that vanish with } y).$$

One must of course remember that our u is specific to Coulomb energies, whereas most prima-facie comparable densities in the literature are meant to describe the distribution either of the total energy, or of the contributions from the transverse Maxwell field.

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